

The electroweak form factor $\hat{\kappa}(q^2)$ and the running of $\sin^2 \hat{\theta}_W$

A. Ferroglia^{1,2,a}, G. Ossola^{3,b}, A. Sirlin^{3,c}

¹ Fakultät für Physik, Universität Freiburg, 79104 Freiburg, Germany

² Institut für Theoretische Teilchenphysik, Universität Karlsruhe, 76128 Karlsruhe, Germany

³ Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA

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Abstract. Gauge independent form factors $\rho^{(e;e)}$ and $\hat{\kappa}^{(e;e)}(q^2)$ for Møller scattering at $s \ll m_W^2$ are derived. It is pointed out that $\hat{\kappa}^{(e;e)}$ is very different from its counterparts in other processes. The relation between the effective parameter $\hat{\kappa}^{(e;e)}(q^2, \mu) \sin^2 \hat{\theta}_W(\mu)$ and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ is derived in a scale-independent manner. A gauge- and process-independent running parameter $\sin^2 \hat{\theta}_W(q^2)$, based on the pinch-technique self-energy $a_{\gamma Z}(q^2)$, is discussed for all q^2 values. At $q^2 = 0$ it absorbs very accurately the Czarnecki–Marciano calculation of the Møller scattering asymmetry at low s values, and at $q^2 = m_Z^2$ it is rather close to $\sin^2 \theta_{\text{eff}}^{\text{lept}}$. The q^2 dependence of $\sin^2 \hat{\theta}_W(q^2)$ is displayed in the space- and time-like domains.

1 Introduction

The form factors ρ and $\kappa(q^2)$ that incorporate the effect of the electroweak corrections in the neutral current amplitudes have played an important role in precision studies of the standard model (SM) [1]. In particular, their effect has been discussed in detail in ν -hadron and ν -lepton scattering at momentum transfers $|q^2| \ll m_W^2$ [2–4], as well as in $e^+ + e^- \rightarrow f + \bar{f}$ near the Z^0 peak [5]. We recall that $\kappa(q^2)$ accompanies the electroweak mixing parameter $\sin^2 \theta_W$ in the Z^0 - $f\bar{f}$ coupling, while ρ multiplies the full neutral current amplitude. A modified version of $\kappa(m_Z^2)$, denoted as $\hat{\kappa}(m_Z^2)$, has also been important in establishing the connection between $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, employed by the Electroweak Working Group (EWWG) to analyze physics at the Z^0 peak, and the $\overline{\text{MS}}$ parameter $\sin^2 \hat{\theta}_W(m_Z)$ [6]. The amplitude $\kappa(q^2)$ and the parameter $\sin^2 \theta_W$ are frequently used in two renormalization schemes: the on-shell framework where $\sin^2 \theta_W \equiv 1 - m_W^2/m_Z^2$ [7], and the $\overline{\text{MS}}$ approach, where one employs $\sin^2 \hat{\theta}_W(\mu)$ and the form factor is denoted as $\hat{\kappa}$. One has, by definition, the relation [3]

$$\kappa(q^2) \sin^2 \theta_W = \hat{\kappa}(q^2, \mu) \sin^2 \hat{\theta}(\mu), \quad (1)$$

where μ is the 't Hooft scale. In the on-shell scheme, $\kappa(q^2)$ and $\sin^2 \theta_W$ are μ -independent and can be separately regarded as physical observables, while in the $\overline{\text{MS}}$ framework it is the combination $\hat{\kappa}(q^2, \mu) \sin^2 \hat{\theta}(\mu)$ that plays that

function. The traditional construction of ρ and $\hat{\kappa}(q^2, \mu)$ at $|q^2| \ll m_W^2$ leads to gauge-invariant expressions, which are, however, process dependent.

Recently, Czarnecki and Marciano [8–10] emphasized that $\hat{\kappa}$ is particularly important in polarized Møller scattering at low s values ($s = (p_1 + p_2)^2$ where p_1 and p_2 are the four momenta of the initial electrons). In fact, at the tree level the asymmetries measured in that process are proportional to $1 - 4 \sin^2 \hat{\theta}_W$, a very small number. Since in the presence of electroweak corrections this factor is replaced at $q^2 = 0$ by $1 - 4\hat{\kappa}^{(e;e)}(0, m_Z) \sin^2 \hat{\theta}_W(m_Z)$, and $\hat{\kappa}^{(e;e)}(0, m_Z) \approx 1.03$, their effect induces a sharp reduction in the predicted asymmetries. This observation is of particular interest at present in view of the proposed E158 fixed target experiment at SLAC [11].

In Sect. 2 we discuss the construction of gauge-independent form factors $\rho^{(e;e)}$ and $\hat{\kappa}^{(e;e)}(q^2, \mu)$ relevant to Møller scattering at s (and therefore q^2) $\ll m_W^2$, and evaluate them in the framework of the general R_ξ gauge. Their comparison with the traditionally defined amplitudes for other processes, such as ν -lepton scattering at $|q^2| \ll m_W^2$, is discussed. The relation between the effective angle defined by $\hat{\kappa}^{(e;e)}(q^2, \mu) \sin^2 \hat{\theta}_W(\mu)$ and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ is analyzed.

In Sect. 3 we discuss a gauge- and process-independent running parameter $\sin^2 \hat{\theta}_W(q^2)$, defined on the basis of the pinch technique (PT) γ - Z self-energy, and show that it approximates very accurately the electroweak corrections for Møller scattering at low s values, while it is rather close to $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ at $q^2 = m_Z^2$. The q^2 dependence of $\sin^2 \hat{\theta}_W(q^2)$ is then illustrated over a large range of values in the space-like and time-like domains.

^a e-mail: andrea.ferroglia@physik.uni-freiburg.de

^b e-mail: giovanni.ossola@physics.nyu.edu

^c e-mail: alberto.sirlin@nyu.edu

2 Gauge-independent form factors $\rho^{(e; e)}$ and $\hat{\kappa}^{(e; e)}(q^2, \mu)$

The traditional approach to obtain the form factors ρ and $\hat{\kappa}$ at $|q^2| \ll m_W^2$ in the case of ν - l and ν -hadron scattering [3, 4] has been to consider the electroweak corrections in the limit in which external fermion masses and momentum transfers are neglected relative to m_W . The electroweak corrections can then be written as expressions bilinear in the matrix elements of J_μ^Z and J_μ^γ , the fermionic currents coupled to Z^0 and γ , respectively. Contributions bilinear in J_μ^Z are absorbed in ρ while those proportional to $J_\mu^Z J_\mu^\gamma$ define the form factor $\hat{\kappa}(q^2)$. Photonic corrections to the external legs are treated separately. Writing the fermionic current of Z^0 in the form

$$J_\mu^Z = \bar{\psi} \frac{C_3 \gamma_\mu a_-}{2} \psi - \sin^2 \hat{\theta}_W J_\mu^\gamma, \quad (2)$$

where $a_- \equiv (1 - \gamma_5)/2$, J_μ^γ is the electromagnetic current and C_3 (± 1) is twice the third component of weak isospin, this procedure leads to the replacement of the tree-level neutral current amplitude M_{ZZ}^0 by

$$M_{ZZ}^0 \rightarrow \frac{im_Z^2}{q^2 - m_Z^2} \frac{8G_F}{\sqrt{2}} \rho^{(i; i')} \langle f | J_\mu^Z | i \rangle \langle f' | J_\mu^Z | i' \rangle, \quad (3)$$

$$J_\mu^Z \rightarrow \bar{\psi} \frac{C_3 \gamma_\mu a_-}{2} \psi - \hat{\kappa}^{(i; i')}(q^2, \mu) \sin^2 \hat{\theta}_W(\mu) J_\mu^\gamma, \quad (4)$$

where $G_F = 1.16637(1) \times 10^{-5} / \text{GeV}^2$ is the Fermi constant determined from μ decay, and the superscripts $(i; i')$ refer to the initial fermions in the process under consideration. Thus, as mentioned in Sect. 1, ρ and $\hat{\kappa}$ are process dependent. In some cases, like ν - l scattering, it is possible to absorb the complete electroweak corrections in these form factors. In other cases, such as ν -hadron scattering, this is not possible since the electroweak corrections induce hadronic currents with an isospin structure not present at the tree level. The latter are then treated as contributions additional to those contained in (3) and (4).

The approach described above has two important virtues:

- (i) the dominant electroweak corrections are incorporated as compact and rather simple modifications of the tree-level neutral current amplitude, and
- (ii) the form factors $\rho^{(i; i')}$ and $\hat{\kappa}^{(i; i')}(q^2, \mu)$ are gauge independent.

In order to implement this construction in the case of Møller scattering, we consider the diagrams of Fig. 1, which contain all the gauge-dependent contributions to the $\hat{\kappa}$ form factor associated with the $\langle f' | J_\mu^Z | i' \rangle$ matrix element. The mirror image of those diagrams contributes to the $\hat{\kappa}$ factor present in the $\langle f | J_\mu^Z | i \rangle$ amplitude.

Using the results of [12], the diagrams of Fig. 1 can be calculated in the general R_ξ gauge. The gauge dependencies of the graphs indeed cancel and, performing the $\overline{\text{MS}}$ subtraction, we find that their contribution to $\hat{\kappa}$ in the case of the Møller scattering at s (and therefore $|q^2| \ll m_W^2$) is given by

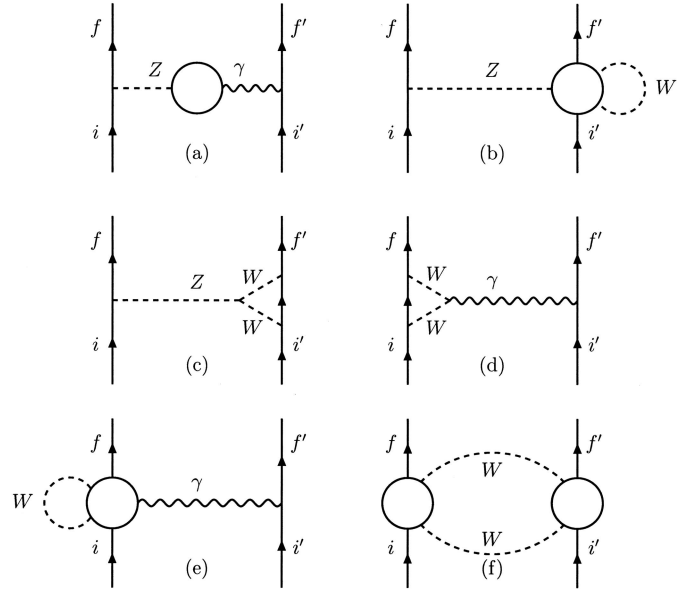


Fig. 1. Feynman diagrams that give gauge-dependent contributions to the $\hat{\kappa}$ form factor in $\langle f' | J_\mu^Z | i' \rangle$. The circle in Fig. 1a represents the contribution of $A_{Z\gamma}$, including its fermionic and bosonic components. The solid line circles in Figs. 1b,e indicate a sum of diagrams in which the ends of the W propagator are attached in all possible ways to the fermion lines. In particular, they include the W contribution to the wave function renormalization of the external lines. The diagram in Fig. 1f represents the W - W box diagrams, whether crossed or uncrossed

$$\begin{aligned} \hat{\kappa}^{(e; e)}(q^2, \mu) = & 1 \\ & + \frac{\alpha}{2\pi\hat{s}^2} \ln\left(\frac{m_Z}{\mu}\right) \left[-\frac{1}{3} \sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) + 7\hat{c}^2 + \frac{1}{6} \right] \\ & + \frac{\alpha}{2\pi\hat{s}^2} \left[-\sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) I_i(q^2) \right. \\ & \left. + \left(\frac{7}{2}\hat{c}^2 + \frac{1}{12} \right) \ln c^2 - \frac{23}{18} + \frac{\hat{s}^2}{3} \right], \quad (5) \end{aligned}$$

where

$$I_i(q^2) = \int_0^1 dx x(1-x) \ln \frac{m_i^2 - q^2 x(1-x)}{m_Z^2}, \quad (6)$$

the i summation is over the fundamental fermions and includes a color factor 3 for quarks, C_i (± 1) is twice the third component of weak isospin for fermion i , Q_i is its electric charge in units of the proton charge e , m_i its mass, $c^2 \equiv m_W^2/m_Z^2$, and $\hat{s}^2 \equiv 1 - \hat{c}^2$ is an abbreviation for the $\overline{\text{MS}}$ parameter $\sin^2 \hat{\theta}_W(m_Z)$.

The corresponding gauge-independent form factor $\rho^{(e; e)}$ contains contributions from several diagrams discussed in [2], and in the Møller scattering case becomes

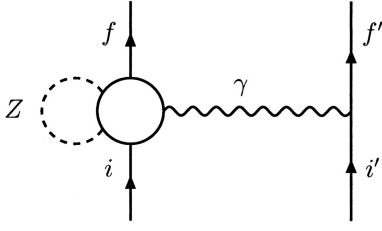


Fig. 2. Gauge-independent vertex diagram. The meaning of the solid line circle is the same as in Fig. 1

$$\rho^{(e;e)} = 1 + \frac{\hat{\alpha}(m_Z)}{4\pi\hat{s}^2} \left\{ \frac{3}{4} \frac{\ln c^2}{\hat{s}^2} - \frac{3}{4} + \frac{3}{4} \frac{m_t^2 (1 + \delta_{\text{QCD}})}{m_W^2} + \frac{3}{4} \xi \left(\frac{\ln(c^2/\xi)}{c^2 - \xi} + \frac{1}{c^2} \frac{\ln \xi}{1 - \xi} \right) \right\}, \quad (7)$$

where $\xi \equiv m_H^2/m_Z^2$, $\hat{\alpha}^{-1}(m_Z) = 127.9 \pm 0.1$, and $\delta_{\text{QCD}} \approx -0.12$ is a QCD correction [13].

Aside from the electroweak corrections in (5) and (7), there are several additional contributions involving Z - Z and γ - Z boxes, the vertex diagram of Fig. 2 and QED corrections. These additional contributions are gauge independent and have been evaluated separately in [8].

At $q^2 = 0$, (5) becomes

$$\hat{\kappa}^{(e;e)}(0, \mu) = 1 + \frac{\alpha}{2\pi\hat{s}^2} \ln \left(\frac{m_Z}{\mu} \right) \left[-\frac{1}{3} \sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) + 7\hat{c}^2 + \frac{1}{6} \right] + \frac{\alpha}{2\pi\hat{s}^2} \left[\frac{1}{3} \sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) \ln \left(\frac{m_Z}{m_i} \right) + \left(\frac{7}{2} \hat{c}^2 + \frac{1}{12} \right) \ln c^2 - \frac{23}{18} + \frac{\hat{s}^2}{3} \right]. \quad (8)$$

In order to evaluate $\hat{\kappa}^{(e;e)}(0, \mu) \sin^2 \hat{\theta}_W(\mu)$ it is convenient to set $\mu = m_Z$. The parameter $\sin^2 \hat{\theta}(m_Z)$ can be obtained from $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ by using the analysis of [6]. Since in Sect. 3 we will consider a running parameter at large q^2 values, for the purpose of this paper we choose not to decouple the top quark in the explicit summations in (5) and (8), or in the definition of $\sin^2 \hat{\theta}(m_Z)$. In that case, for $m_t = 174.3$ GeV, we have $\sin^2 \hat{\theta}(m_Z) = \sin^2 \theta_{\text{eff}}^{\text{lept}}/1.00044$. For $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ we will use the central value of the current world average, 0.23148 [14], which leads to $\sin^2 \hat{\theta}(m_Z) = 0.23138$. In the appendix we report the results obtained if, instead, one employs as input the central value derived from the leptonic asymmetries, namely $\sin^2 \hat{\theta}(m_Z)_{(l)} = 0.23103$. For the contribution of the first five flavors of quarks to the i summation in (8) one must invoke dispersion relations and experimental data on $e^+ e^- \rightarrow$ hadrons: we use a recent update by Marciano [15]. Further employing $m_W = 80.426$ GeV, $m_Z = 91.1875$ GeV, $m_t = 174.3$ GeV [14], $m_H = 200$ GeV, we obtain

$$\hat{\kappa}^{(e;e)}(0, m_Z) = 1.0270 \pm 0.0025, \quad (9)$$

$$\rho^{(e;e)} = 1.0034. \quad (10)$$

In the region $115 \text{ GeV} \leq m_H \leq 200 \text{ GeV}$, $\rho^{(e;e)}$ varies slowly with m_H . For instance, $\rho^{(e;e)} = 1.0037$ at $m_H = 115$ GeV.

Most of the difference between (7)–(10) and the results reported in [8] is due to the fact that we have retained the contributions of the W - W boxes in $\hat{\kappa}^{(e;e)}(0, m_Z)$ and $\rho^{(e;e)}$ in order to ensure their gauge independence. In contrast, in [8] these contributions have been separated from the form factors in a particular gauge, namely the 't Hooft–Feynman gauge. In this gauge, the W - W boxes contribute $\alpha/4\pi\hat{s}^2$ to $\rho^{(e;e)}$ and $-\alpha/4\pi\hat{s}^2$ to $\hat{\kappa}^{(e;e)}$. However, in the general R_ξ gauges, they may be arbitrarily different. A second, smaller difference, is that, as explained before, we have included the top quark contribution in (8). We have already pointed out that the Z - Z boxes are gauge independent and therefore they may be separated without affecting the gauge properties of $\hat{\kappa}^{(e;e)}$ and $\rho^{(e;e)}$. They are suppressed by a factor $1 - 4\hat{s}^2$ and our calculation of these terms agrees with that reported in [8]. The contribution of the Z - γ boxes, the diagram of Fig. 2, and QED corrections are also proportional to $1 - 4\hat{s}^2$ and are contained in a function $F_1(y, Q^2)$ ($y = Q^2/s$; $Q^2 = -q^2$) evaluated in [8]. As an interesting illustration, it is pointed out in that paper that $F_1(1/2, 0.02 \text{ GeV}^2) = -0.0041 \pm 0.0010$.

Putting together the values of the form factors evaluated in the present paper (see (9) and (10)), the Z - Z box diagrams and the calculation of $F_1(y, Q^2)$ reported in [8], one finds that the overall effect of the electroweak corrections is to replace

$$1 - 4 \sin^2 \hat{\theta}_W(m_Z) \rightarrow \rho^{(e;e)} \left\{ 1 - 4 \hat{\kappa}^{(e;e)}(0, m_Z) \sin^2 \hat{\theta}_W(m_Z) + (Z - Z)_{\text{box}} + F_1(y, Q^2) \right\}, \quad (11)$$

which at $Q^2 = 0.025 \text{ GeV}^2$ and $y = 1/2$ equals $0.0454 \pm 0.0023 \pm 0.0010$. This is numerically very close to the result obtained by Czarnecki and Marciano because these authors chose to separate the contributions of the W - W boxes in the 't Hooft–Feynman gauge, where they are reasonably small (cf. Sect. 4). As emphasized in [8], since $1 - 4 \sin^2 \hat{\theta}_W(m_Z) = 0.07448 \pm 0.00068$, the effect of the electroweak corrections in this case is to reduce the asymmetries by $\approx 39\%$! Clearly, the bulk of the reduction is contained in $1 - 4 \hat{\kappa}^{(e;e)}(0, m_Z) \sin^2 \hat{\theta}_W(m_Z) = 0.0495$. Detailed studies of radiative corrections to polarized Møller scattering at low and high energies are given in [16, 17], respectively.

It should be pointed out that the correction $\hat{\kappa}^{(e;e)}(0, m_Z) - 1 = 0.0270$ is very different from the corresponding effects in other processes. For instance, in $\nu_\mu - e$ and $\nu_e - e$ scattering one obtains $\hat{\kappa}^{(\nu_\mu; e)}(0, m_Z) - 1 = -0.0032$ and $\hat{\kappa}^{(\nu_e; e)}(0, m_Z) - 1 = -0.0210$, respectively [4]. The large difference is mainly due to sizable ‘‘charge radius’’ diagrams that contribute negatively and to significant and negative W - W and Z - Z box contributions. In contrast, the vertex diagram of Fig. 2 as well as the Z - Z

boxes are suppressed by $1 - 4\hat{s}^2$ in the Møller scattering case, while the large corrections associated with the γ - Z self-energy are not.

Using the gauge-independent form factor $\hat{\kappa}^{(e;e)}(q^2, \mu)$, it is possible to define an effective electroweak parameter for $|q^2| \ll m_W^2$:

$$\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2) \equiv \text{Re} \hat{\kappa}^{(e;e)}(q^2, \mu) \sin^2 \hat{\theta}_W(\mu). \quad (12)$$

For $q^2 < 0$, $\hat{\kappa}^{(e;e)}(q^2, \mu)$ is real and the Re instruction is not necessary. Dividing (12) by

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \text{Re} \hat{\kappa}(m_Z^2, \mu) \sin^2 \hat{\theta}_W(\mu), \quad (13)$$

where $\hat{\kappa}(m_Z^2, \mu)$ is the form factor discussed in [6], and neglecting two-loop effects not enhanced by powers of m_t^2 , we find

$$\begin{aligned} \sin^2 \theta_{\text{eff}}^{(e;e)}(q^2) & \quad (14) \\ & = \left\{ 1 + \text{Re} \left[\hat{\kappa}^{(e;e)}(q^2, \mu) - \hat{\kappa}(m_Z^2, \mu) \right] \right\} \sin^2 \theta_{\text{eff}}^{\text{lept}}. \end{aligned}$$

The μ dependence cancels in (14) so that we may choose $\mu = m_Z$, and (14) becomes

$$\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2) = \left[\text{Re} \hat{\kappa}^{(e;e)}(q^2, m_Z) - 0.00044 \right] \sin^2 \theta_{\text{eff}}^{\text{lept}}. \quad (15)$$

Equation (15) establishes a scale-independent relationship between the two gauge-independent parameters $\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2)$ and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, which may be regarded as physical observables.

It is worthwhile to point out that the form factors $\hat{\kappa}(q^2, \mu)$ and $\hat{\kappa}(m_Z^2, \mu)$ are quite different conceptually even when $\hat{\kappa}$ is evaluated at $q^2 = m_Z^2$. While $\hat{\kappa}$ is relevant to four-fermion scattering processes, $\hat{\kappa}(m_Z^2, m_Z)$ involves the decay amplitude of an on-shell Z^0 into an $l\bar{l}$ pair.

3 The running of $\sin^2 \hat{\theta}_W$

In this section we discuss the possibility of constructing a running electroweak mixing parameter for arbitrary values of q^2 . We will impose four theoretical requirements:

- (i) it should be process independent;
- (ii) since $\sin^2 \hat{\theta}_W$ is related to γ - Z mixing, it should involve the $A_{\gamma Z}(q^2)$ self-energy in a fundamental way;
- (iii) it should be gauge independent, at least in the class of R_ξ gauges;
- (iv) it should be as simple as possible.

At first sight, these requirements seem difficult to satisfy. In fact, we have seen in Sect. 2 that, in order to obtain gauge-independent form factors, the contributions of box diagrams must be included. Since in general box diagrams depend on two kinematic variables, s and q^2 , and are process dependent, it is not trivial to see how to achieve our aims. However, there is a well-known framework that allows us to satisfy these conditions, namely the pinch technique (PT) [18–27]. We recall that the PT is a prescription

that judiciously combines the conventional self-energies with “pinch parts” from vertex and box diagrams in such a manner that the new self-energies are gauge independent and possess desirable theoretical properties. In [22], it was shown that the “pinch parts” can be identified with amplitudes involving appropriate equal-time commutators of currents, which explains why they are process independent and unaffected by strong interaction dynamics.

In this section we discuss a running electroweak mixing parameter $\sin^2 \hat{\theta}_W(q^2)$ defined in terms of the PT γZ self-energy. Specifically,

$$\sin^2 \hat{\theta}_W(q^2) \equiv \left(1 - \frac{\hat{c}}{\hat{s}} \frac{a_{\gamma Z}(q^2, \mu)}{q^2} \right) \sin^2 \hat{\theta}_W(\mu), \quad (16)$$

where $a_{\gamma Z}(q^2, \mu)$, the PT γZ self-energy of the SM, can be conveniently expressed as [22]

$$\begin{aligned} a_{\gamma Z}(q^2, \mu) & \quad (17) \\ & = A_{\gamma Z}(q^2, \mu)|_{\xi_W=1} - \frac{2e^2}{\hat{c}\hat{s}} (2q^2 \hat{c}^2 - m_W^2) I_{WW}(q^2, \mu). \end{aligned}$$

In (17) $A_{\gamma Z}(q^2, \mu)|_{\xi_W=1}$ is the conventional γ - Z self-energy evaluated in the 't Hooft–Feynman gauge $\xi_W = 1$, and

$$I_{WW}(q^2, \mu) = \frac{1}{16\pi^2} \int_0^1 dx \ln \left[\frac{m_W^2 - q^2 x(1-x) - i\epsilon}{\mu^2} \right]. \quad (18)$$

Very simple analytic formulae for $I_{WW}(q^2, \mu)$ are given in (A5)–(A7) of [22]. In (18) we have performed the $\overline{\text{MS}}$ subtraction of $\delta = (n-4)^{-1} + (\gamma - \ln 4\pi)/2$. It is understood that the same subtraction has been implemented in $A_{\gamma Z}(q^2, \mu)|_{\xi_W=1}$.

Since the RHS of (16) is process and gauge independent, it satisfies our theoretical requirements. It is also important to remember that, unlike $A_{\gamma Z}(0, \mu)|_{\xi_W=1}$, $a_{\gamma Z}(0, \mu) = 0$, so that (16) is regular as $q^2 \rightarrow 0$.

It is convenient to define

$$\hat{\kappa}^{\text{PT}}(q^2, \mu) = 1 - \frac{\hat{c}}{\hat{s}} \frac{a_{\gamma Z}(q^2, \mu)}{q^2}. \quad (19)$$

In the range $|q^2| \ll m_W^2$, we find

$$\begin{aligned} \hat{\kappa}^{\text{PT}}(q^2, \mu) & = 1 \\ & + \frac{\alpha}{2\pi\hat{s}^2} \ln \left(\frac{m_Z}{\mu} \right) \left[-\frac{1}{3} \sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) + 7\hat{c}^2 + \frac{1}{6} \right] \\ & + \frac{\alpha}{2\pi\hat{s}^2} \left[-\sum_i (C_i Q_i - 4\hat{s}^2 Q_i^2) I_i(q^2) \right. \\ & \left. + \left(\frac{7}{2} \hat{c}^2 + \frac{1}{12} \right) \ln c^2 - \frac{\hat{c}^2}{3} \right]. \quad (20) \end{aligned}$$

Comparing (20) with (5), we have, for $|q^2| \ll m_W^2$:

$$\hat{\kappa}^{\text{PT}}(q^2, \mu) = \hat{\kappa}^{(e;e)}(q^2, \mu) + \frac{17}{18} \frac{\alpha}{2\pi\hat{s}^2}. \quad (21)$$

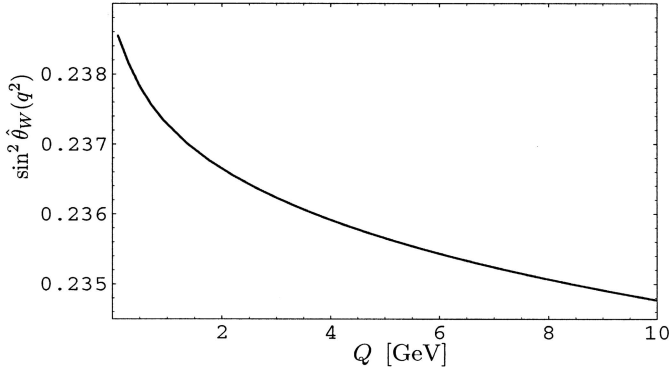


Fig. 3. $\sin^2 \hat{\theta}_W(q^2)$ as a function of $Q = (-q^2)^{1/2}$ in the space-like domain $q^2 < 0$, for $0 \leq Q \leq 10$ GeV

Thus, while $\hat{\kappa}^{(e;e)}(0, m_Z) = 1.0270 \pm 0.0025$ (cf. (9)),

$$\hat{\kappa}^{\text{PT}}(0, m_Z) = 1.0317 \pm 0.0025, \quad (22)$$

a difference of 4.7×10^{-3} . We see that the PT form factor approximates rather well $\hat{\kappa}^{(e;e)}(0, m_Z)$ at $q^2 = 0$. More interestingly, the PT running parameter evaluated at $q^2 = 0$ almost exactly absorbs the complete calculation reported in (11) for $y = 1/2$ and $Q^2 = 0.025 \text{ GeV}^2$. In fact, using (16), (19) and (22), we have

$$\sin^2 \hat{\theta}_W(0) = 0.2387 \pm 0.0006. \quad (23)$$

This leads to

$$1 - 4 \sin^2 \hat{\theta}_W(0) = 0.0452 \pm 0.0023, \quad (24)$$

in very close agreement with $0.0454 \pm 0.0023 \pm 0.0010$, reported after (11), when all electroweak corrections are taken into account. Of course, this very accurate agreement will generally not hold for other values of y and Q^2 , but it is interesting that $1 - 4 \sin^2 \hat{\theta}_W(0)$ does absorb quite precisely the bulk of the corrections to Møller scattering at $s \ll m_W^2$.

In order to evaluate $\sin^2 \hat{\theta}_W(q^2)$ as a function of q^2 , we set $\mu = m_Z$ and employ the expressions for $A_{\gamma Z}^{(f)}(q^2, m_Z)$ from [2], $A_{\gamma Z}^{(b)}(q^2, m_Z)|_{\xi_W=1}$ from [28] (f and b mean fermionic and bosonic contributions), and $I_{WW}(q^2, m_Z)$ from [22]. As for α , we follow the approach of [29] and replace $\alpha^{-1} \rightarrow \alpha^{-1}(q^2)$,

$$\alpha^{-1}(q^2) = \alpha^{-1} - \frac{2}{\pi} \sum_i Q_i^2 \int_0^1 dx x (1-x) \ln \left\{ \frac{m_i^2 - q^2 x (1-x)}{m_i^2} \right\}. \quad (25)$$

In evaluating $A_{\gamma Z}^{(f)}(q^2, m_Z)$ and $\alpha^{-1}(q^2)$, we employ the effective light quark masses and the QCD correction factor discussed in [15].

Figures 3 and 4 show $\sin^2 \hat{\theta}_W(q^2)$ in the space-like domain $q^2 < 0$, appropriate for e^-e^- colliders, as a function of $Q = (-q^2)^{1/2}$. Table 1 gives a few representative values of $\hat{\kappa}(q^2, m_Z)$ and $\sin^2 \hat{\theta}_W(q^2)$ in that region.

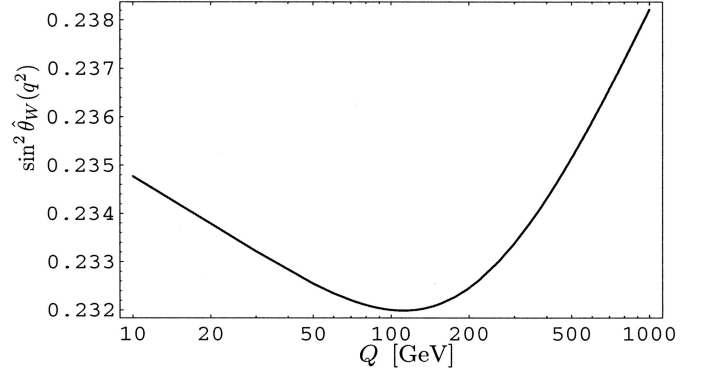


Fig. 4. Same as in Fig. 3, for $10 \text{ GeV} \leq Q \leq 1 \text{ TeV}$

Table 1. $\hat{\kappa}^{\text{PT}}(q^2, m_Z)$ and $\sin^2 \hat{\theta}_W(q^2)$ for $q^2 < 0$ at different values of Q ($Q = \sqrt{-q^2}$)

Q [GeV]	$\hat{\kappa}^{\text{PT}}(q^2, m_Z)$	$\sin^2 \hat{\theta}_W(q^2)$
0	1.0317	0.2387
m_Z	1.0028	0.2320
111	1.0026	0.2320
500	1.0163	0.2352
1000	1.0296	0.2382

We see that $\sin^2 \hat{\theta}_W(q^2)$ equals 0.2387 at $Q = 0$, 0.2320 at $Q = m_Z$, reaches a minimum of 0.23199 at $Q = 111 \text{ GeV}$, and then increases monotonically to 0.2352 at $Q = 500 \text{ GeV}$ and 0.2382 at $Q = 1 \text{ TeV}$.

Figure 4 bears a close resemblance to a curve presented in [9, 10] for a running parameter constructed on the basis of the diagrams in Figs. 1a,d. The theoretical foundation of the two running parameters is, however, very different. While the PT self-energy is gauge independent within the class of R_ξ gauges, the sum of the diagrams in Figs. 1a,d is not. Thus, in principle, by varying the gauge in the second approach one can alter the values and Q^2 dependence of the running parameter.

A second problem, already discussed in [22], is that the diagram in Fig. 1d is not truly process independent, even in the limit of neglecting the external fermion masses. For instance, when the external fermion is a quark or a hadron, there are QCD corrections not present in the leptonic case. As explained in [22], this problem is neatly bypassed in the PT approach since the pinch part of Fig. 1d is unaffected by strong interaction dynamics.

In the time-like domain $q^2 > 0$, $\hat{\kappa}^{\text{PT}}(q^2, \mu)$ is complex and we define the running parameter as

$$\sin^2 \hat{\theta}_W(q^2) \equiv \text{Re} [\hat{\kappa}^{\text{PT}}(q^2, m_Z)] \sin^2 \hat{\theta}_W(m_Z). \quad (26)$$

In Figs. 5 and 6 we present values of $\sin^2 \hat{\theta}_W(q^2)$ in the time-like domain $q^2 > 0$, appropriate to e^+e^- colliders, as a function of $Q = (q^2)^{1/2}$. A sharp decrease associated with the W - W threshold is very visible. In order to soften the behavior in that region we have included the W width by means of the replacement $m_W^2 \rightarrow m_W^2 - i m_W \Gamma_W$,

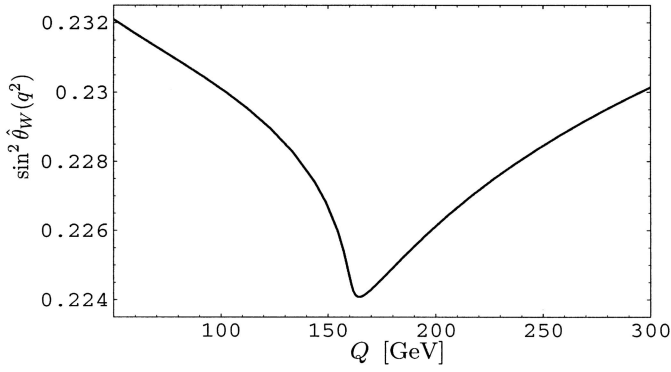


Fig. 5. $\sin^2 \hat{\theta}_W(q^2)$ as a function of $Q = (q^2)^{1/2}$ in the time-like domain $q^2 > 0$, for $50 \leq Q \leq 300$ GeV

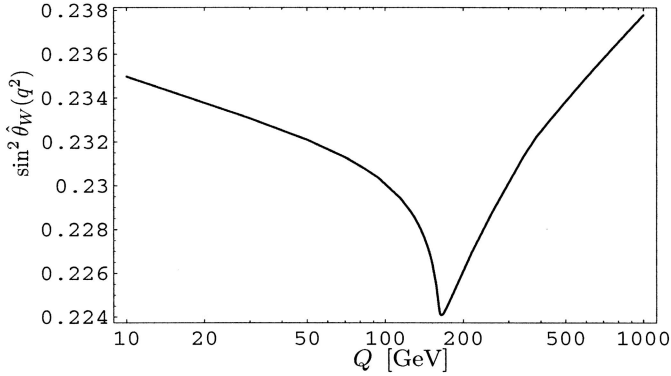


Fig. 6. Same as in Fig. 5, for $10 \text{ GeV} \leq Q \leq 1 \text{ TeV}$

Table 2. $\hat{\kappa}^{\text{PT}}(q^2, m_Z)$ and $\sin^2 \hat{\theta}_W(q^2)$ for $q^2 > 0$ at different values of Q ($Q = \sqrt{q^2}$)

Q [GeV]	$\hat{\kappa}^{\text{PT}}(q^2, m_Z)$	$\sin^2 \hat{\theta}_W(q^2)$
m_Z	0.9961	0.2305
164	0.9685	0.2241
500	1.0107	0.2338
1000	1.0277	0.2378

with $m_{2W} = m_W / (1 + (\Gamma_W/m_W)^2)^{1/2} = 80.398$ GeV and $\Gamma_{2W} = m_{2W} \Gamma_W / m_W = 2.117$ GeV [30].

At $Q = m_Z$, we find $\sin^2 \hat{\theta}_W(m_Z^2) = 0.23048$, which is lower than $\sin^2 \theta_{\text{eff}}^{\text{lept}} = 0.23148$ by 0.43%. Although not in perfect consonance, the two parameters are rather close. Other representative values of $\hat{\kappa}^{\text{PT}}$ and $\sin^2 \hat{\theta}_W(q^2)$ in the time-like region are given in Table 2. We see that $\sin^2 \hat{\theta}_W(q^2)$ ranges from 0.2387 at $Q = 0$ to 0.2305 at $Q = m_Z$, reaches a minimum of 0.2241 at $Q = 164$ GeV, and then increases monotonically to 0.2338 at $Q = 500$ GeV and 0.2378 at $Q = 1$ TeV.

It is interesting to note that $\sin^2 \hat{\theta}_W(-m_Z^2)$ is slightly larger than $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, the difference being 0.22%. Thus, $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ lies between $\sin^2 \hat{\theta}_W(m_Z^2)$ and $\sin^2 \hat{\theta}_W(-m_Z^2)$.

4 Discussion

Aside from the general observation that physical results in gauge theories should be parametrized in a gauge-independent manner, there are specific reasons why this is particularly important in the case of the electroweak form factors ρ and $\hat{\kappa}$.

(i) If $\hat{\kappa}^{(e;e)}(q^2, \mu)$ is gauge independent, the effective electroweak mixing parameter $\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2)$ defined in (12) is also gauge independent, and consequently may be regarded as a physical observable. In particular, $\sin^2 \theta_{\text{eff}}^{(e;e)}(0)$ can be measured by polarized Møller scattering with considerable precision. If $\hat{\kappa}^{(e;e)}(q^2, \mu)$ is defined in a gauge-dependent manner, this is theoretically unfounded, since $\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2)$ would not qualify as a physical observable.

(ii) The parameterization in terms of ρ and $\hat{\kappa}$ involves a factorization of one-loop electroweak corrections (see, for example, (3), (4) and (11)). If the form factors are defined in a gauge-dependent manner, one can make the two-loop effects induced by the factorization to vary arbitrarily by simply changing the gauge used to calculate the form factors. These effects can be very large for large values of the gauge parameter and, in fact, the calculation diverges in the unitary, i.e. the physical, gauge!

(iii) It should be also pointed out that the electroweak form factors for other processes, such as discussed in [2–4], have been defined in a gauge-independent manner.

In order to circumvent these theoretical problems, in Sect. 2 we have discussed the derivation of gauge-independent form factors $\rho^{(e;e)}$ and $\hat{\kappa}^{(e;e)}(q^2, \mu)$ appropriate to Møller scattering at $s \ll m_Z^2$. For the reasons explained after (11), the overall electroweak corrections including these form factors, as well as the gauge-independent corrections that have been separated out, agree numerically very closely with the results of [8]. We have pointed out that $\hat{\kappa}^{(e;e)}(q^2, \mu) - 1$ is quite different from the corresponding form factors in other processes such as $\nu_\mu - e$ or $\nu_e - e$ scattering. Thus, it is not possible, even at $|q^2| \ll m_W^2$, to find a universal electroweak mixing parameter that absorbs the bulk of the electroweak corrections in all processes.

However, as emphasized in [8–10], experiments on polarized Møller scattering are very special in that the asymmetries measured in that process are greatly affected by $\hat{\kappa}^{(e;e)}(q^2, m_Z)$ and may be used to measure the effective mixing parameter $\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2)$. At $s \ll m_W^2$ this is of considerable present interest in view of the proposed E158 experiment at SLAC [11].

In Sect. 2 we have also derived a scale-independent relation between $\sin^2 \theta_{\text{eff}}^{(e;e)}(q^2)$ and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$. In particular, this relation may be employed to discuss the electroweak corrections to Møller scattering in the effective scheme of renormalization, in which residual scale dependencies cancel in finite orders of perturbation theory, and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ plays the role of the basic electroweak parameter [31–34].

In Sect. 3 we have discussed a gauge- and process-independent running parameter based on the PT γZ self-

energy [22]. At $q^2 = 0$ it absorbs very precisely the electroweak corrections to Møller scattering evaluated in [8] at $y = 1/2$ and $Q^2 = 0.025 \text{ GeV}^2$, and at $q^2 = m_Z^2$ it lies within 0.43% of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$. Thus it provides an attractive theoretical and phenomenological framework to describe the running of the electroweak mixing parameter over a large range of q^2 values.

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Appendix

In this appendix we report the numerical results obtained if one employs as input the central value of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ derived from the leptonic asymmetries, namely $\sin^2 \theta_{\text{eff}}^{\text{lept}} = 0.23113$, rather than the world average.

Without decoupling the top quark contributions, we have $\sin^2 \hat{\theta}_W(m_Z) = 0.23103$. The value in (9) is replaced by 1.0271 ± 0.0025 , while (11) equals $0.0467 \pm 0.0023 \pm 0.0010$. $1 - 4 \hat{\kappa}^{(e;e)}(0, m_Z) \sin^2 \hat{\theta}_W(m_Z)$ becomes 0.0508, to be compared with $1 - 4 \sin^2 \hat{\theta}_W(m_Z) = 0.0759$. Equations (22), (23) and (24) equal 1.0319 ± 0.0025 , 0.2384 ± 0.0006 , and 0.0464 ± 0.0023 , respectively. The values of $\sin^2 \hat{\theta}_W(q^2)$ are smaller by 0.0003 or 0.0004 than those in Table 1 and by 0.0004 than those in Table 2.

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¹ This paper contains a discussion of previous attempts to construct effective running couplings for the electroweak sector of the SM.